A note on the relation between temporally-increasing and spatially-increasing disturbances in hydrodynamic stability

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The frequency and amplification rates for a disturbance growing with respect to time are compared with those of a spatially-growing wave having the same wave-number. For small rates of amplification it is shown that the frequencies are equal to a high order of approximation, and that the spatial growth is related to the time growth by the group velocity.

The solution of the Orr–Sommerfeld equation at a fixed Reynolds number yields a characteristic function relating the wave-number, frequency and amplification rate of the disturbance. The perturbation stream function in two-dimensional parallel flows has the form

$$\psi(x, y, t) = \phi(y)\, e^{i(\alpha x-\beta t)}, \tag{1}$$

where α and β are in general complex and α_r (real part of α) is the wave-number, β_r the frequency, and α_i, β_i the amplification rates. ($\beta_i > 0$ denotes amplification in time, while $\alpha_i < 0$ denotes spatially amplified disturbances in the positive direction of x.)

The characteristic function relating the eigenvalues α and β at a given Reynolds number is

$$F(\alpha, \beta) = 0. \tag{2}$$

The two cases of most interest are those in which either α or β is wholly real. In the first case (T), when α is real, we have a time-dependent amplification system and in the second case (S), when β is real, a spatially-dependent system of the type discussed by Watson (1962). It will be shown here that simple relations exist between the parameters arising in these two special cases. It should be noted that these relations do not constitute a transformation of a time-dependent solution into a spatially-dependent one, but merely provide a link between the values of the parameters existing in the two problems.

The two special cases labelled (T) and (S) are:

Case (T): time-dependent, $\alpha_i(\mathrm{T}) = 0$,

$$\alpha = \alpha_r(\mathrm{T}), \quad \beta = \beta_r(\mathrm{T}) + i\beta_i(\mathrm{T}). \tag{3}$$

Case (S): spatially-dependent, $\beta_i(\mathrm{S}) = 0$,

$$\alpha = \alpha_r(\mathrm{S}) + i\alpha_i(\mathrm{S}), \quad \beta = \beta_r(\mathrm{S}). \tag{4}$$

Let us now consider a general problem involving both time and spatially growing modes in some given region, in which β is assumed to be an analytic function of α. With this assumption the Cauchy–Riemann relations hold,

$$\partial\beta_r/\partial\alpha_r = \partial\beta_i/\partial\alpha_i, \quad \partial\beta_r/\partial\alpha_i = -\partial\beta_i/\partial\alpha_r. \tag{5}$$

If we now integrate these relations with respect to α_i from state (T) to state (S), keeping $\alpha_r = \text{const.} = \alpha_r(\text{T})$, then

$$\beta_i(\text{T}) = -\int_0^{\alpha_i(\text{S})} \frac{\partial\beta_r}{\partial\alpha_r}\,d\alpha_i, \tag{6}$$

and

$$\beta_r(\text{S}) - \beta_r(\text{T}) = -\int_0^{\alpha_i(\text{S})} \frac{\partial\beta_i}{\partial\alpha_r}\,d\alpha_i. \tag{7}$$

Since α_r is constant between (T) and (S) we must have $\alpha_r(\text{S}) = \alpha_r(\text{T})$.

Shen's (1954) calculations, which apply to state (T) where α_i is zero, show that for Poiseuille and Blasius flow $\partial\beta_i/\partial\alpha_r = O(\beta_{im})$, where β_{im} is the maximum value of β_i for the given Reynolds number. His paper also shows that $\beta_{im} = O(10^{-3})$. We also expect α_i in state (S) to be small, of order β_i, and that $\partial\alpha_i/\partial\beta_r = O(\beta_{im})$. It seems reasonable therefore to extend the argument to all states between (T) and (S) so that the integrand in (7) is $O(\beta_{im})$. It then follows that

$$\beta_r(\text{S}) = \beta_r(\text{T}) + O\{\beta_{im}\alpha_i(\text{S})\}. \tag{8}$$

Neglecting terms of order β_{im}^2, we have to a very good approximation

$$\beta_r(\text{S}) = \beta_r(\text{T}) \tag{9}$$

within the entire neutral curve.

If we expand $\partial\beta_r/\partial\alpha_r$ in a Taylor series about any point α_i^* in the range 0 to $\alpha_i(\text{S})$ and substitute in (6), we get

$$\beta_i(\text{T}) = -\alpha_i(\text{S})\frac{\partial\beta_r}{\partial\alpha_r}(\alpha_i^*) - \{\tfrac{1}{2}\alpha_i^2(\text{S}) - \alpha_i(\text{S})\,\alpha_i^*\}\frac{\partial^2\beta_r}{\partial\alpha_r\partial\alpha_i}(\alpha_i^*) + \ldots.$$

But from (5), $\partial\beta_r/\partial\alpha_i = -\partial\beta_i/\partial\alpha_r$, so that

$$\frac{\beta_i(\text{T})}{\alpha_i(\text{T})} = -\frac{\partial\beta_r}{\partial\alpha_r}(\alpha_i^*) + O\{\alpha_i(\text{S})\,\beta_{im}\}. \tag{10}$$

Again neglecting terms of order β_{im}^2 we have, provided that $\partial\beta_r/\partial\alpha_r$ is non-zero,

$$\beta_i(\text{T})\,/\alpha_i(\text{S}) = -\partial\beta_r/\partial\alpha_r, \tag{11}$$

where $\partial\beta_r/\partial\alpha_r$ can be evaluated at any station between (T) and (S).

A similar analysis can also be carried out if the frequency is kept constant and it is assumed that α is an analytic function of β. It can then be shown that, at least to $O(\beta_{im}^2)$, the wave-number remains constant during the transformation and that the ratio of temporal to spatial growth factor is $-(\partial\alpha_r/\partial\beta_r)$. This result is consistent with (11) for we have:

$$\frac{\partial\alpha}{\partial\beta} = \frac{1}{\partial\beta/d\alpha} = \frac{1}{(\partial\beta_r/\partial\alpha_r) + i(\partial\beta_i/\partial\alpha_r)} = \frac{(\partial\beta_r/\partial\alpha_r) - i(\partial\beta_i/\partial\alpha_r)}{(\partial\beta_r/\partial\alpha_r)^2 + (\partial\beta_i/\partial\alpha_r)^2}$$

and

$$\frac{d\alpha}{d\beta} = \frac{\partial\alpha_r}{\partial\beta_r} + i\frac{\partial\alpha_i}{\partial\beta_r}$$

giving

$$\left(\frac{\partial\alpha_r}{\partial\alpha_r}\right)_{\beta_i} = \left(\frac{\partial\beta_r}{\partial\alpha_r}\right)_{\alpha i}^{-1} + O(\beta_{im}^2).$$

We have to order (β_{im}^2) at most

$$\alpha_r(\mathrm{T}) = \alpha_r(\mathrm{S}), \quad \beta_r(\mathrm{T}) = \beta_r(\mathrm{S}), \quad \text{and} \quad \beta_i(\mathrm{T})/\alpha_i(\mathrm{S}) = -\partial\beta_r/\partial\alpha_r. \tag{12}$$

It is worth noting that it is possible to obtain the amplification factors for three-dimensional spatially growing waves by using a transformation of the type discussed in this note. Squire's (1933) tranformation can be used to transform time-dependent two-dimensional solutions to three-dimensional ones. After performing this transformation the solution for three-dimensional spatially growing waves is obtained by the use of a similar transformation to that discussed in this note.

REFERENCES

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